A spectral approach to the Dirac equation in the non-extreme Kerr–Newmann metric

2009 J. Phys. A: Math. Theor. 42 295204

(http://iopscience.iop.org/1751-8121/42/29/295204)

J. Phys. A: Math. Theor. **42** (2009) 295204 (15pp) doi:10.1088/1751-8113/42/29/295204

A spectral approach to the Dirac equation in the non-extreme Kerr–Newmann metric

Monika Winklmeier[1] **and Osanobu Yamada**[2]

[1] Departamento de Matemáticas, Universidad de los Andes, A.A. 4976 Bogotá, Columbia
[2] Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525–8577, Japan

E-mail: mwinklme@uniandes.edu.co and yamadaos@se.ritsumei.ac.jp

Received 21 January 2009, in final form 18 May 2009
Published 30 June 2009
Online at stacks.iop.org/JPhysA/42/295204

Abstract
We investigate the local energy decay of solutions of the Dirac equation in the non-extreme Kerr–Newman metric. First, we write the Dirac equation as a Cauchy problem and define the Dirac operator. It is shown that the Dirac operator is selfadjoint in a suitable Hilbert space. With the RAGE theorem, we show that for each particle its energy located in any compact region outside the event horizon of the Kerr–Newman black hole decays in the time mean.

PACS numbers: 02.30.Sa, 03.65.Nk, 04.70.Bw

1. Introduction

The Kerr–Newman metric is the most general stationary solution of Einstein's field equations and has the physical interpretation of a massive charged rotating black hole. The physical parameters M, Q and $a = J/M$ have the interpretation as its mass, electric charge and angular momentum parameter.

As in flat spacetime, a spin-$\frac{1}{2}$ particle is described by a Dirac equation. In the Kerr–Newman metric, the Dirac equation is given by the coupled system of partial differential equations (see, e.g., Page 1976, Chandrasekhar 1998)

$$(\widehat{\mathfrak{R}} + \widehat{\mathfrak{A}})\widehat{\Psi} = 0 \tag{1}$$

where

$$\widehat{\mathfrak{R}} := \begin{pmatrix} imr & 0 & \sqrt{\Delta}\mathfrak{R}_+^{t,\varphi} & 0 \\ 0 & -imr & 0 & \sqrt{\Delta}\mathfrak{R}_-^{t,\varphi} \\ \sqrt{\Delta}\mathfrak{R}_-^{t,\varphi} & 0 & -imr & 0 \\ 0 & \sqrt{\Delta}\mathfrak{R}_+^{t,\varphi} & 0 & imr \end{pmatrix},$$

$$\widehat{\mathfrak{A}} := \begin{pmatrix} -\mathfrak{D} & 0 & 0 & \mathfrak{L}_+^{t,\varphi} \\ 0 & \mathfrak{D} & -\mathfrak{L}_-^{t,\varphi} & 0 \\ 0 & \mathfrak{L}_+^{t,\varphi} & -\mathfrak{D} & 0 \\ -\mathfrak{L}_-^{t,\varphi} & 0 & 0 & \mathfrak{D} \end{pmatrix}$$

and

$$\mathfrak{D} := am\cos\theta, \tag{2}$$

$$\mathfrak{R}_\pm^{t,\varphi} := \frac{\partial}{\partial r} \pm \frac{\mathrm{i}}{\Delta}\left[(r^2+a^2)\,\mathrm{i}\frac{\partial}{\partial t} + a\,\mathrm{i}\frac{\partial}{\partial\varphi} + eQr\right] \qquad \text{on } (r_+,\infty), \tag{3}$$

$$\mathfrak{L}_\pm^{t,\varphi} := \frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} \mp \left[a\sin\theta\,\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{\sin\theta}\,\mathrm{i}\frac{\partial}{\partial\varphi}\right] \qquad \text{on } (0,\pi). \tag{4}$$

The physical parameters m and e are the mass and the electric charge of the spin-$\frac{1}{2}$-particle which is described by the wavefunction $\widehat{\Psi}$.

The functions Δ and Σ are given by

$$\Delta(r) = r^2 - 2Mr + a^2 + Q^2 = (r-M)^2 + a^2 + Q^2 - M^2,$$
$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta.$$

According to the black hole parameters M, Q and a, three cases can arise: if $a^2+Q^2-M^2 < 0$, then the function Δ has exactly two distinct zeros; this case is called the *non-extreme Kerr–Newman metric*. If $a^2+Q^2-M^2 = 0$, then the function Δ has exactly one zero; this case is referred to as the *extreme Kerr–Newman metric*. In both cases the metric is interpreted physically as the spacetime generated by a charged rotating massive black hole. If $a^2+Q^2-M^2 > 0$, then the function Δ has no zeros; in this case the Kerr–Newman metric has no interpretation as the metric of a black hole.

It is a remarkable fact that the Dirac equation (1) can be separated into ordinary differential equations. This was first shown by (Chandrasekhar 1976). The first step is to separate the dependence on the coordinates t and φ by the ansatz

$$\widehat{\Psi}(r,\theta,\varphi,t) = \mathrm{e}^{-\mathrm{i}\omega t}\widetilde{\Psi}(r,\theta,\varphi) \tag{5}$$

for some $\omega \in \mathbb{R}$. We call ω an *energy eigenvalue* of equation (1) if it has a solution of the form (5) which is square integrable in the sense explained below in section 2.

The separation of the Dirac equation into ordinary differential equations is achieved by the ansatz

$$\widehat{\Psi}(r,\theta,\varphi,t) = \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}\kappa\varphi}\begin{pmatrix} X_-(r)S_+(\theta) \\ X_+(r)S_-(\theta) \\ X_+(r)S_+(\theta) \\ X_-(r)S_-(\theta) \end{pmatrix}$$

with $\kappa \in \mathbb{Z}+\frac{1}{2}$. One obtains two ordinary differential equations, the so-called radial equation for the radial coordinate r and the so-called angular equation for the angular coordinate θ. Both equations can be written as eigenvalue problems for selfadjoint operators in appropriate Hilbert spaces. The spectrum of the corresponding angular operator consists of simple and discrete eigenvalues which are unbounded from below and above (see, e.g., Winklmeier 2006). Belgiorno and Martellini (1999) showed that the spectrum of the radial equation comprises all of the real axis. Recently, it was shown that in the non-extreme Kerr–Newman metric no embedded eigenvalues exist. In the extreme Kerr–Newman case, however, embedded eigenvalues can exist (see Schmid 2004, Winklmeier and Yamada 2006).

In this paper we investigate the local energy decay of solutions of the Dirac equation (1). We assume that the non-extreme case holds, that is, there are no energy eigenvalues. To prove the local energy decay we do not employ the separation ansatz described above, but we rewrite the Dirac equation as a Cauchy problem. After some transformations, we obtain the equation

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(x,\theta,\varphi,t) = \mathcal{S}(x,\theta)^{-1}\mathfrak{H}\Psi(x,\theta,\varphi,t) \tag{6}$$

where $\mathcal{S}(x,\theta)$ is a bounded and boundedly invertible 4×4-matrix and $\mathfrak{H}$ is the formal Dirac operator associated with the Dirac equation (1). We show that the expression $\mathfrak{H}$ on the right-hand side has a selfadjoint realization H in an appropriate Hilbert space (theorem 2.1). To the best of our knowledge this is the first mathematically rigorous proof of the selfadjointness of the Dirac operator in the Kerr–Newman metric[3]. The operator H can be written as the sum over partial wave operators $H^{(\kappa)}$, $\kappa\in\mathbb{Z}+\frac{1}{2}$. In order to apply the RAGE theorem to the Dirac operator, we show in theorem 3.3 that it enjoys a Rellich type property. One of our main results in this paper is theorem 4.2 which shows the local energy decay of the wavefunctions in the time mean: let $U^{(\kappa)}$ be the group associated with the skew-selfadjoint operator $-\mathrm{i}\mathcal{S}^{-1}H^{(\kappa)}$. Then, any function f which is subjected to the time evolution given by (6) satisfies

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-R}^{R}\int_{0}^{\pi}\|U^{(\kappa)}(t)f(x,\theta)\|^2_{\mathbb{C}^4}\mathrm{d}x\,\mathrm{d}\theta\right]\mathrm{d}t = 0$$

for every $R>0$. Since the expression in the square brackets is related to the energy of the particle described by f at time t in the region $\Omega_0 := (-R,R)\times(0,\pi)$, this result is related to the decay of the particle's energy in Ω_0. The above result remains valid if the bounded electric potential eQr given by the background metric is substituted by a possibly unbounded potential q satisfying

$$q\in C^1((r_+,\infty)),\qquad \lim_{r\to\infty}\frac{q(r)}{r}\ \text{exists},\qquad q'(r)=O(1)\quad\text{as}\quad r\to\infty.$$

The local energy decay has also been investigated by Finster *et al* (2003). They restrict the Dirac operator to an annulus outside the event horizon and impose Dirichlet type boundary conditions so that the restricted operator has purely discrete spectrum. Hence the propagator associated with it is a sum of projections. Extending the radii of the annulus to the event horizon and infinity, respectively, the sum of the propagator representation becomes an integral. However, they do not show that their results are independent of the chosen boundary conditions.

Our approach deals directly with the Dirac operator in the exterior of the black hole. Using the fact that the partial wave operators $H^{(\kappa)}$ have no point spectrum, the local energy decay of wavefunctions with initially compact support follows by methods from scattering theory. In particular, we do not impose any additional boundary conditions.

2. Transformation of the Dirac equation to a Cauchy problem

To investigate the time evolution of solutions of the time-dependent Dirac equation we rewrite it as a Cauchy problem.

In the following, we use the Pauli matrices to abbreviate notation:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

[3] After this work was completed we were made aware of Belgiorno and Cacciatori (2008). Therein (theorem 1) a result about essential selfadjointness similar to our theorem 2.1 is proved.

In addition, let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \Sigma_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \qquad j = 1, 2, 3.$$

For the φ-coordinate we always assume the boundary condition $\Psi(r, \theta, \varphi) = \Psi(r, \theta, \varphi + 2\pi)$.

In the remainder of the paper we use the notation

$$\begin{aligned} \Omega_3 &:= (-\infty, \infty) \times (0, \pi) \times [0, 2\pi), \\ \Omega_2 &:= (-\infty, \infty) \times (0, \pi). \end{aligned}$$

Theorem 2.1. *Assume that the function Δ has at least one real zero (that is, either the extreme or the non-extreme Kerr–Newman case holds) and denote the largest zero by r_+. Define the new radial coordinate $x \in (-\infty, \infty)$ by*

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta(r)}. \tag{7}$$

To keep notation simple, we often write r instead $r(x)$.

A function $\widehat{\Psi}$ is a solution of the Dirac equation (1) if and only if

$$\mathrm{i}\frac{\partial}{\partial t}\Psi = \mathcal{S}^{-1}\mathfrak{H}\Psi, \tag{8}$$

where

$$\begin{aligned}
&\Psi(x, \theta, \varphi, t) = (\sin\theta)^{-1/2}\widehat{\Psi}(r(x), \theta, \varphi, t), \\
&\mathcal{S} = I + \frac{a\sqrt{\Delta}\sin\theta}{r^2 + a^2}\begin{pmatrix} \sigma_2 & \\ & -\sigma_2 \end{pmatrix}, \\
&\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2, \\
&\mathfrak{H}_1 = \begin{pmatrix} \mathfrak{h}_1 & \\ & -\mathfrak{h}_1 \end{pmatrix}, \qquad \mathfrak{h}_1 = -\sigma_3 \mathrm{i}\frac{\partial}{\partial x} - \frac{\sqrt{\Delta}}{r^2 + a^2}\left(\sigma_1 \mathrm{i}\frac{\partial}{\partial\theta} + \sigma_2\frac{1}{\sin\theta}\,\mathrm{i}\frac{\partial}{\partial\varphi}\right), \\
&\mathfrak{H}_2 = -\frac{1}{r^2 + a^2}\begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix}\left(a\,\mathrm{i}\frac{\partial}{\partial\varphi} + \mathrm{e}Qr\right) \\
&\qquad - \frac{mr\sqrt{\Delta}}{r^2 + a^2}\begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix} + \frac{am\sqrt{\Delta}\cos\theta}{r^2 + a^2}\begin{pmatrix} & \mathrm{i}I_2 \\ -\mathrm{i}I_2 & \end{pmatrix}.
\end{aligned} \tag{9}$$

Moreover, the operator H_0 defined by

$$H_0\Psi = \mathfrak{H}\Psi, \qquad \mathcal{D}(H_0) = \mathscr{C}_0^\infty(\Omega_3)^4 \tag{10}$$

is essentially selfadjoint in the Hilbert space $\mathscr{L}^2(\Omega_3)^4$ and the operator

$$\mathcal{S}^{-1}H_0$$

is essentially selfadjoint in the Hilbert space

$$\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4 := \mathscr{L}^2(\Omega_3)^4 \qquad \textit{with scalar product } (\cdot,\cdot)_{\mathcal{S}}$$

given by

$$(\Psi, \Phi)_{\mathcal{S}} = (\Psi, \mathcal{S}\Phi) = \int_{\mathbb{R}}\int_0^\pi\int_0^{2\pi}\langle\Psi(x, \theta, \varphi), \mathcal{S}(x, \theta)\Phi(x, \theta, \varphi)\rangle_{\mathbb{C}^4}\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}x;$$

accordingly, the norm on $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)$ will be denoted by $\|\cdot\|_{\mathcal{S}}$.

Let H be the closure of H_0. Then $\mathcal{S}^{-1}H$ is the closure of $\mathcal{S}^{-1}H_0$, and we call $\mathcal{S}^{-1}H$ the *time-independent Dirac operator* in the Kerr–Newman metric.

Remark 2.2. Observe that for each $(x, \theta) \in (-\infty, \infty) \times (0, \pi)$ the matrix $\mathcal{S}(x, \theta)$ is bounded and boundedly invertible since

$$\left| \frac{a\sqrt{\Delta}\sin\theta}{r(x)^2 + a^2} \right| \leqslant \frac{|a|r(x)}{r(x)^2 + a^2} \leqslant \frac{1}{2}.$$

Therefore, the norms on $\mathscr{L}^2(\Omega_3)^4$ and $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$ are equivalent.

Proof of theorem 2.1. First we rearrange equation (1) such that all time derivatives are on the left-hand side and all other terms are on the right-hand side, thus we obtain

$$\left[\begin{pmatrix} & -\mathrm{i}\sigma_3 \\ \mathrm{i}\sigma_3 & \end{pmatrix} + \frac{a\sqrt{\Delta}\sin\theta}{r^2 + a^2} \begin{pmatrix} & \sigma_1 \\ \sigma_1 & \end{pmatrix} \right] \mathrm{i}\frac{\partial}{\partial t}\Psi(x, \theta, \varphi, t)$$

$$= \left[\begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix} \frac{\partial}{\partial x} + \frac{\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} & \mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & \end{pmatrix} \frac{\partial}{\partial\theta} + \frac{\sqrt{\Delta}}{(r^2 + a^2)\sin\theta} \begin{pmatrix} & -\sigma_1 \\ -\sigma_1 & \end{pmatrix} \mathrm{i}\frac{\partial}{\partial\varphi} \right.$$

$$+ \frac{1}{r^2 + a^2} \begin{pmatrix} & \mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_3 & \end{pmatrix} \left(a\,\mathrm{i}\frac{\partial}{\partial\varphi} + \mathrm{e}Qr \right) + \frac{\mathrm{i}mr\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} \sigma_3 & \\ & -\sigma_3 \end{pmatrix}$$

$$\left. + \frac{am\sqrt{\Delta}\cos\theta}{r^2 + a^2} \begin{pmatrix} -\sigma_3 & \\ & -\sigma_3 \end{pmatrix} \right] \widehat{\Psi}(x, \theta, \varphi, t).$$

Multiplication from the left by $\left(\begin{smallmatrix} & -\mathrm{i}\sigma_3 \\ \mathrm{i}\sigma_3 & \end{smallmatrix}\right)$ yields

$$\left[I + \frac{a\sqrt{\Delta}\sin\theta}{r^2 + a^2} \begin{pmatrix} \sigma_2 & \\ & -\sigma_2 \end{pmatrix} \right] \mathrm{i}\frac{\partial}{\partial t}\Psi(x, \theta, \varphi, t)$$

$$= \left[\begin{pmatrix} -\sigma_3 & \\ & \sigma_3 \end{pmatrix} \mathrm{i}\frac{\partial}{\partial x} + \frac{\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} -\sigma_1 & \\ & \sigma_1 \end{pmatrix} \mathrm{i}\frac{\partial}{\partial\theta} \right.$$

$$+ \frac{\sqrt{\Delta}}{(r^2 + a^2)\sin\theta} \begin{pmatrix} -\sigma_2 & \\ & \sigma_2 \end{pmatrix} \mathrm{i}\frac{\partial}{\partial\varphi}$$

$$- \frac{1}{r^2 + a^2} \begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix} \left(a\,\mathrm{i}\frac{\partial}{\partial\varphi} + \mathrm{e}Qr \right) - \frac{mr\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}$$

$$\left. + \frac{am\sqrt{\Delta}\cos\theta}{r^2 + a^2} \begin{pmatrix} & \mathrm{i}I_2 \\ -\mathrm{i}I_2 & \end{pmatrix} \right] \Psi(x, \theta, \varphi, t)$$

$$= (\mathfrak{H}_1 + \mathfrak{H}_2)\Psi(x, \theta, \varphi, t),$$

thus the first assertion of the theorem is proved.

By remark 2.2 the norms on $\mathscr{L}^2(\Omega_3)^4$ and $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$ are equivalent. Hence, H_0 is essentially selfadjoint in $\mathscr{L}^2(\Omega_3)^4$ if and only if $\mathcal{S}^{-1}H_0$ is so in $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$. Therefore, it suffices to show that H_0 is essentially selfadjoint.

Since $\frac{\partial}{\partial\varphi}$ with the boundary condition stated at the beginning of this section has the complete system of eigenfunctions $(\mathrm{e}^{-\mathrm{i}\kappa\varphi})_{\kappa\in\mathbb{Z}+1/2}$ and all the differential expressions in the equation above commute with $\frac{\partial}{\partial\varphi}$, each solution Ψ has a representation

$$\Psi(x,\theta,\varphi,t) = \sum_{\kappa\in\mathbb{Z}+1/2} \mathrm{e}^{-\mathrm{i}\kappa\varphi}\Psi^{(\kappa)}(x,\theta,t).$$

Ψ satisfies (8) if and only if each $\Psi^{(\kappa)}$ satisfies

$$\mathrm{i}\frac{\partial}{\partial t}\Psi^{(\kappa)} = \mathcal{S}^{-1}\mathfrak{H}^{(\kappa)}\Psi^{(\kappa)}, \tag{11}$$

where $\mathfrak{H}^{(\kappa)}$, $\mathfrak{H}_1^{(\kappa)}$ and $\mathfrak{H}_2^{(\kappa)}$ are obtained from $\mathfrak{H}$, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ by substituting $\mathrm{i}\frac{\partial}{\partial\varphi}$ with κ. Moreover, H_0 is essentially selfadjoint if and only if each partial wave operator $H_0^{(\kappa)}$ (the restriction of H_0 to the partial wave space represented by $\mathrm{e}^{-\mathrm{i}\kappa\varphi}$) is essentially selfadjoint.

In proposition 2.4 below, it is shown that $h_{1,0}^{(\kappa)}$, defined by

$$h_{1,0}^{(\kappa)}\Psi^{(\kappa)} = \mathfrak{h}_1^{(\kappa)}\Psi^{(\kappa)}, \qquad \mathcal{D}\big(h_{1,0}^{(\kappa)}\big) = \mathscr{C}_0^\infty(\Omega_2)^2, \tag{12}$$

is essentially selfadjoint in $\mathscr{L}^2(\Omega_2)^2$ with the usual scalar product.

Since both $\mathfrak{H}_1$ and $\mathcal{S}^{-1}$ are block diagonal and $\mathfrak{H}_2^{(\kappa)}$ defines a bounded selfadjoint operator on $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)^4$, the assertion on the essential selfadjointness of $H_0^{(\kappa)}$, $\kappa\in\mathbb{Z}+\frac{1}{2}$, and therefore of H_0 follows. □

Lemma 2.3. *For $\kappa\in\mathbb{Z}+\frac{1}{2}$ let*

$$\mathcal{A}_\kappa := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{\partial}{\partial\theta} + \frac{\kappa}{\sin\theta}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and define the unitary matrix

$$W := \begin{pmatrix}0 & 1\\ \mathrm{i} & 0\end{pmatrix}.$$

The operator $\mathcal{A}_\kappa$ is the angular part of the Dirac equation (1) arising from Chandrasekhar's separation process in the case $a=0$.

(i) For each $\kappa\in\mathbb{Z}+1/2$, the operator $\mathcal{A}_\kappa$ with domain $\big(\mathscr{C}_0^\infty(0,\pi)\big)^2$ is essentially selfadjoint in $\mathscr{L}^2((0,\pi),\mathrm{d}\theta)^2$. We denote its closure again by $\mathcal{A}_\kappa$. It is compactly invertible and its spectrum consists of simple eigenvalues only, given by

$$\lambda_m^\kappa := \operatorname{sign}(m)\big(|\kappa| - \tfrac{1}{2} + |m|\big), \qquad m=\pm1,\pm2,\ldots.$$

We denote the corresponding normalized eigenfunctions by g_m^κ.

(ii) For every $m\in\mathbb{Z}\setminus\{0\}$ we have $\lambda_{-m}^\kappa = -\lambda_m^\kappa$ and $\sigma_3 g_m^\kappa = -g_{-m}^\kappa$.

(iii) $W\mathcal{A}_\kappa W^{-1} = W^{-1}\mathcal{A}_\kappa W = \mathrm{i}\sigma_3\mathcal{A}_{-\kappa}$.

(iv) $W^{-1}\mathfrak{h}_1^{(\kappa)}W = W^{-1}\sigma_3 W\Big(-\mathrm{i}\frac{\partial}{\partial x}\Big) - \frac{\sqrt{\Delta}}{r^2+a^2}\mathcal{A}_\kappa = \sigma_3\Big(\mathrm{i}\frac{\partial}{\partial x}\Big) - \frac{\sqrt{\Delta}}{r^2+a^2}\mathcal{A}_\kappa$.

Proof. For (i), we refer to Winklmeier (2006). Statements (iii) and (iv) are easily verified by direct computation. To prove (ii) we observe that all eigenvalues of $\mathcal{A}_\kappa$ are simple and that

$$\mathcal{A}_\kappa\sigma_3 g_m^\kappa = \sigma_3\sigma_3\mathcal{A}_\kappa\sigma_3 g_m^\kappa = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathcal{A}_\kappa\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}g_m^\kappa$$

$$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(-\mathcal{A}_\kappa)g_m^\kappa = -\lambda_m^\kappa\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}g_m^\kappa = -\lambda_m^\kappa\sigma_3 g_m^\kappa = \lambda_{-m}^\kappa\sigma_3 g_m^\kappa.$$ □

Proposition 2.4. *The minimal operator $h_{1,0}^{(\kappa)}$ defined in (12) by*

$$h_{1,0}^{(\kappa)}\psi = \mathfrak{h}_1^{(\kappa)}\psi = \left(-\sigma_3 \mathrm{i}\frac{\partial}{\partial r} - \frac{\sqrt{\Delta}}{r^2+a^2}\left(\sigma_1 \mathrm{i}\frac{\partial}{\partial\theta} + \sigma_2\frac{1}{\sin\theta}\,\mathrm{i}\frac{\partial}{\partial\varphi}\right)\right)\psi,$$
$$\mathcal{D}\big(h_{1,0}^{(\kappa)}\big) = \mathscr{C}_0^\infty((-\infty,\infty)\times(0,\pi))^2,$$

is essentially selfadjoint in the space $\mathscr{L}^2(\Omega_2)^4$.

Proof. Obviously, $\mathfrak{h}_1^{(\kappa)}$ is Hermitian and $h_{1,0}^{(\kappa)}$ is symmetric. Hence, the lemma is proved if we have shown that $\ker\big(h_{1,0}^{(\kappa)*} \pm \mathrm{i}\big) = \{0\}$. The equality $h_{1,0}^{(\kappa)*}\psi = \pm\mathrm{i}\psi$ is equivalent to

$$\big[W^{-1}h_{1,0}^{(\kappa)*}W\big]W^{-1}\psi = \pm\mathrm{i}W^{-1}\psi$$

with the matrix W as defined in lemma 2.3.

If we set $\varphi := W^{-1}\psi$, then, by lemma 2.3, the equation above is equivalent to

$$\left[\sigma_3\left(-\mathrm{i}\frac{\partial}{\partial x}\right) + \frac{\sqrt{\Delta}}{r^2+a^2}\mathcal{A}_\kappa\right]\varphi = \pm\mathrm{i}\varphi(x,\theta). \tag{13}$$

Since the set of eigenfunctions of the operator $\mathcal{A}_\kappa$, denoted by g_n^κ as in lemma 2.3, is complete in $\mathscr{L}^2((0,\pi),\mathrm{d}\theta)^2$, we can expand the function φ in (13) as

$$\varphi(x,\theta) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \xi_m(x)g_m^\kappa(\theta)$$

with functions $\xi_m \in \mathscr{L}^2((-\infty,\infty),\mathrm{d}x)$. Thus (13) yields

$$\begin{aligned}\pm\mathrm{i}\sum_{m\in\mathbb{Z}\setminus\{0\}}\xi_m(x)g_m(\theta) &= \sum_{m\in\mathbb{Z}\setminus\{0\}} -\mathrm{i}\frac{\partial}{\partial x}\sigma_3\xi_m(x)g_m^\kappa(\theta) + \sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{\sqrt{\Delta}}{r^2+a^2}\xi_m(x)\mathcal{A}_\kappa g_m^\kappa(\theta)\\ &= \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathrm{i}\frac{\partial}{\partial x}\xi_m(x)g_{-m}(\theta) + \frac{\lambda_m^\kappa\sqrt{\Delta}}{r^2+a^2}\xi_m(x)g_m^\kappa(\theta).\end{aligned}$$

Taking the scalar product with $g_\mu(\theta)$ in $\mathbb{C}^2$ and integrating with respect to θ yields on the left-hand side

$$\pm\mathrm{i}\int_0^\pi \sum_{m\in\mathbb{Z}\setminus\{0\}}\xi_m\big\langle g_m^\kappa(\theta)g_\mu(\theta)\big\rangle_{\mathbb{C}^2}\,\mathrm{d}\theta = \pm\mathrm{i}\sum_{m\in\mathbb{Z}\setminus\{0\}}\xi_m\int_0^\pi\big\langle g_m^\kappa(\theta)g_\mu(\theta)\big\rangle_{\mathbb{C}^2}\,\mathrm{d}\theta = \pm\mathrm{i}\xi_\mu$$

and on the right-hand side

$$\begin{aligned}&\int_0^\pi\left\langle\sum_{m\in\mathbb{Z}\setminus\{0\}}\mathrm{i}\frac{\partial}{\partial x}\xi_m(x)g_{-m}(\theta) + \frac{\lambda_m^\kappa\sqrt{\Delta}}{r^2+a^2}\xi_m(x)g_m^\kappa(\theta),\,g_\mu(\theta)\right\rangle_{\mathbb{C}^2}\mathrm{d}\theta\\ &\quad= \sum_{m\in\mathbb{Z}\setminus\{0\}}\mathrm{i}\frac{\partial}{\partial x}\xi_m(x)\int_0^\pi\langle g_{-m}(\theta),g_\mu(\theta)\rangle_{\mathbb{C}^2}\,\mathrm{d}\theta\\ &\qquad+ \sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{\lambda_m^\kappa\sqrt{\Delta}}{r^2+a^2}\xi_m(x)\int_0^\pi\langle g_m(\theta),g_\mu(\theta)\rangle_{\mathbb{C}^2}\,\mathrm{d}\theta\int_0^\pi\langle g_m(\theta),g_\mu(\theta)\rangle_{\mathbb{C}^2}\,\mathrm{d}\theta\\ &\quad= \mathrm{i}\frac{\partial}{\partial x}\xi_{-m}(x) + \frac{\lambda_m^\kappa\sqrt{\Delta}}{r^2+a^2}\xi_m(x).\end{aligned}$$

Hence we obtain

$$\pm\ \mathrm{i}\xi_\mu(x) = \mathrm{i}\frac{\partial}{\partial x}\xi_{-\mu}(x) + \frac{\lambda_\mu^\kappa\sqrt{\Delta}}{r^2+a^2}\xi_\mu(x). \tag{14}$$

An analogous procedure with $g_{-\mu}$ gives

$$\pm \mathrm{i}\xi_{-\mu}(x) = \mathrm{i}\frac{\partial}{\partial x}\xi_{\mu}(x) + \frac{\lambda_{\mu}^{\kappa}\sqrt{\Delta}}{r^2+a^2}\xi_{-\mu}(x). \tag{15}$$

Combining equations (14) and (15) we obtain

$$\pm\,\mathrm{i}\begin{pmatrix}\xi_{\mu}\\ \xi_{-\mu}\end{pmatrix} = \begin{pmatrix}\frac{\lambda_{\mu}^{\kappa}\sqrt{\Delta}}{r^2+a^2} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} & -\frac{\lambda_{\mu}^{\kappa}\sqrt{\Delta}}{r^2+a^2}\end{pmatrix}\begin{pmatrix}\xi_{\mu}\\ \xi_{-\mu}\end{pmatrix}. \tag{16}$$

The multiplication operator $\frac{\lambda_{\mu}^{\kappa}\sqrt{\Delta}}{r^2+a^2}$ is bounded on $\mathscr{L}^2((-\infty,\infty),\mathrm{d}x)$ and it is well known that the operator $\begin{pmatrix}0 & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} & 0\end{pmatrix}$ with domain $\mathscr{C}_0^{\infty}(-\infty,\infty)^2$ is essentially selfadjoint in $\mathscr{L}^2((-\infty,\infty),\mathrm{d}x)^2$, see, for instance, Weidmann (1987, chapter 6.B). Hence the same is true for $\begin{pmatrix}\frac{\lambda_{m}^{\kappa}\sqrt{\Delta}}{r^2+a^2} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\\ \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} & -\frac{\lambda_{\mu}^{\kappa}\sqrt{\Delta}}{r^2+a^2}\end{pmatrix}$ and equation (16) has only the trivial solution in $\mathscr{L}^2((-\infty,\infty),\mathrm{d}x)$. This, however, implies that also equation (13) has only the trivial solution and the lemma is proved. □

Let $H^{(\kappa)}$ be the unique selfadjoint extension of $\mathfrak{H}^{(\kappa)}$ (see (11)). The following result is proved in (Belgiorno and Martellini 1999, Schmid 2004, Winklmeier and Yamada 2006).

Proposition 2.5. *For each half-integer κ, the selfadjoint operator $\mathcal{S}^{-1}H^{(\kappa)}$ in $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)^2$ has no eigenvalues and $\sigma(H^{(\kappa)}) = \sigma_{\mathrm{ess}}(H^{(\kappa)}) = \mathbb{R}$.*

Proof. Let $\Psi(x,\theta)$ be an eigenfunction of $\mathcal{S}^{-1}H^{(\kappa)}$ such that $\mathcal{S}^{-1}H^{(\kappa)}\Psi = \omega\Psi$. Then $\Psi(x,\theta)$ satisfies (1) by replacing $\mathrm{i}\frac{\partial}{\partial t}, \mathrm{i}\frac{\partial}{\partial\varphi}$ by ω,κ, respectively. Therefore $\Psi = 0$ follows from the separation ansatz and theorems IV.2 and IV.5 in (Winklmeier and Yamada 2006), see also (Schmid 2004). □

The above proposition is closely related to the nonexistence of time-periodic solutions. Indeed, assume that $\Psi = \Psi(x,\theta)$ is an eigenfunction of $\mathcal{S}^{-1}H^{(\kappa)}$. Then $\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}\kappa\varphi}\Psi(x,\theta)$ is a periodic solution of (1). However, it was shown (Finster *et al* 2000, Winklmeier and Yamada 2006) that (1) has no non-trivial eigenfunctions, hence $\Psi = 0$.

3. Rellich property

Definition 3.1. *Let Ω be an open set in $\mathbb{R}^n$ and T a selfadjoint operator on $\mathscr{L}^2(\Omega)$. We say that T has the Rellich property, if the following is true: let $F \subset \mathcal{D}(T)$ be such that there exists a positive constant K such that*

$$\|f\|^2 + \|Tf\|^2 \leqslant K, \qquad f \in F.$$

Then F is precompact in $L^2_{loc}(\Omega')$ for every bounded open set $\Omega' \subset \Omega$, that is, every sequence $(f_n)_n \subset F$ has a subsequence $(f_{n_k})_k$ which converges strongly in $\mathscr{L}^2(\Omega')$. (As usual, we identify functions $f \in F$ with the corresponding functions $f|_{\Omega'} \in \mathscr{L}^2(\Omega')$.)

The classical result that differential operators in $\mathscr{L}^2(\mathbb{R}^d)$ have the Rellich property is cited in the appendix.

The aim of this section is to show that $\mathfrak{H}^{(\kappa)}$ has the Rellich property. First we prove the following technical lemma needed for theorem 3.3.

Lemma 3.2. *Let W and $\mathcal{A}_\kappa$, $\kappa \in \mathbb{Z} + \frac{1}{2}$, be defined as in lemma 2.3 and let $H^{(\kappa)}$ be the Dirac operator in the κth partial wave space (see theorem 2.1).*

Then there exists a positive constant K such that

$$\int_{\mathbb{R}} \int_0^\pi \left[\sum_{j=1}^{2} \left\| \frac{\partial}{\partial x} \Psi_j(x,\theta) \right\|_{\mathbb{C}^2}^2 + \frac{\Delta}{(r^2+a^2)^2} \|\mathcal{A}_\kappa \Psi_j(x,\theta)\|_{\mathbb{C}^2}^2 \right] \mathrm{d}\theta\, \mathrm{d}x \tag{17}$$

$$\leqslant K \left(\|\mathcal{S}^{-1} H^{(\kappa)} \Psi\|_{\mathcal{S}}^2 + \|\Psi\|_{\mathcal{S}}^2 \right) \tag{18}$$

for all $\Psi = {}^t(\Psi_1, \Psi_2) \in \mathcal{D}(\mathcal{S}^{-1}H^{(\kappa)})$ where $\Psi_j \in \mathscr{L}_{\mathcal{S}}^2(\Omega_2)^2$, $j = 1, 2$.

Proof. First assume $\Psi \in \mathscr{C}_0^\infty(\Omega_2)^4$. Observe that

$$\frac{1}{2} \leqslant \left\| \left(I + \frac{a \sin\theta \sqrt{\Delta}}{r^2+a^2} \beta \Sigma_2 \right)^{-1} \right\| \leqslant 2, \qquad (r,\theta) \in (-\infty, \infty) \times (0, \pi).$$

With the notation of section 2 we write $\mathcal{S}^{-1}H^{(\kappa)}$ as the sum of the unbounded operator $\mathcal{S}^{-1}H_1^{(\kappa)}$ and the bounded operator $\mathcal{S}^{-1}H_2^{(\kappa)}$ and obtain

$$\begin{aligned} \|\mathcal{S}^{-1}H^{(\kappa)}\Psi\|_{\mathcal{S}}^2 &\geqslant \frac{1}{2} \left\|\mathcal{S}^{-1}H_1^{(\kappa)}\Psi\right\|_{\mathcal{S}}^2 - \left\|\mathcal{S}^{-1}H_2^{(\kappa)}\Psi\right\|_{\mathcal{S}}^2 \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_0^\pi \left\|\mathcal{S}^{-1/2} H_1^{(\kappa)} \Psi(x,\theta)\right\|_{\mathbb{C}^4}^2 \mathrm{d}x\, \mathrm{d}\theta - \left\|\mathcal{S}^{-1}H_2^{(\kappa)}\Psi\right\|_{\mathcal{S}}^2 \\ &\geqslant \frac{1}{4} \int_{\mathbb{R}} \int_0^\pi \left\|H_1^{(\kappa)}\Psi(x,\theta)\right\|_{\mathbb{C}^4}^2 \mathrm{d}x\, \mathrm{d}\theta - \left\|\mathcal{S}^{-1}H_2^{(\kappa)}\right\|^2 \|\Psi\|_{\mathcal{S}}^2. \end{aligned} \tag{19}$$

Recall that

$$\begin{aligned} H_1^{(\kappa)} &= \begin{pmatrix} \sigma_3 & \\ & -\sigma_3 \end{pmatrix} \left(-\mathrm{i} \frac{\partial}{\partial x} \right) + \frac{\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} W^{-1}\mathcal{A}_\kappa W & \\ & -W^{-1}\mathcal{A}_\kappa W \end{pmatrix} \\ &= \begin{pmatrix} \sigma_3 & \\ & -\sigma_3 \end{pmatrix} \left[\left(-\mathrm{i} \frac{\partial}{\partial x} \right) + \frac{\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} \mathrm{i}\mathcal{A}_\kappa & \\ & -\mathrm{i}\mathcal{A}_\kappa \end{pmatrix} \right] \end{aligned}$$

by lemma 2.3. Since the first matrix in the above representation is unitary, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^\pi \left\|H_1^{(\kappa)}\Psi(x,\theta)\right\|_{\mathbb{C}^2}^2 \mathrm{d}x\, \mathrm{d}\theta \\ &\quad = \int_{\mathbb{R}} \int_0^\pi \left\| -\mathrm{i}\frac{\partial}{\partial x}\Psi(x,\theta) + \frac{\sqrt{\Delta}}{r^2+a^2} \begin{pmatrix} \mathrm{i}\mathcal{A}_\kappa & \\ & \mathrm{i}\mathcal{A}_\kappa \end{pmatrix} \Psi(x,\theta) \right\|_{\mathbb{C}^4}^2 \mathrm{d}\theta\, \mathrm{d}x \\ &\quad = \sum_{j=1}^{2} \int_{\mathbb{R}} \int_0^\pi \left\| -\mathrm{i}\frac{\partial}{\partial x}\Psi_j(x,\theta) + \frac{\sqrt{\Delta}}{r^2+a^2}\, \mathrm{i}\mathcal{A}_\kappa \Psi_j(x,\theta) \right\|_{\mathbb{C}^2}^2 \mathrm{d}\theta\, \mathrm{d}x \\ &\quad = \sum_{j=1}^{2} \int_{\mathbb{R}} \int_0^\pi \left[\left\| \frac{\partial}{\partial x}\Psi_j(x,\theta) \right\|_{\mathbb{C}^2}^2 + \frac{\Delta}{(r^2+a^2)^2} \|\mathcal{A}_\kappa \Psi_j(x,\theta)\|_{\mathbb{C}^2}^2 \right] \mathrm{d}\theta\, \mathrm{d}x \\ &\qquad + \int_{\mathbb{R}} \int_0^\pi \left\langle \Psi_j(x,\theta), \left(\frac{\partial}{\partial x} \frac{\sqrt{\Delta}}{r^2+a^2} \right) \mathcal{A}_\kappa \Psi_j(x,\theta) \right\rangle_{\mathbb{C}^2} \mathrm{d}\theta\, \mathrm{d}x \end{aligned} \tag{20}$$

where in the last step we have used that $-\mathrm{i}\partial_x$ is symmetric and $\mathcal{A}_\kappa$ commutes with $-\mathrm{i}\partial_x$ and performed integration by parts. There are no boundary terms since Ψ has compact support.

There exists a positive constant C such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sqrt{\Delta}}{r^2+a^2}\right| = \left|\frac{\mathrm{d}r}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\sqrt{\Delta}}{r^2+a^2}\right| = \frac{\Delta}{r^2+a^2}\left|\frac{\mathrm{d}}{\mathrm{d}r}\frac{\sqrt{\Delta}}{r^2+a^2}\right|$$
$$= \frac{\sqrt{\Delta}}{r^2+a^2}\left|-\frac{2r\Delta}{(r^2+a^2)^2}+\frac{r-M}{r^2+a^2}\right| \leqslant C\frac{\sqrt{\Delta}}{r^2+a^2} \tag{21}$$

since the term in $|\cdot|$ is continuous and tends to zero for $x\to\pm\infty$. Moreover, for every $\varepsilon>0$

$$\left\langle \Psi_j, \left(\frac{\partial}{\partial x}\frac{\sqrt{\Delta}}{r^2+a^2}\right)\mathcal{A}_\kappa\Psi_j\right\rangle_{\mathscr{L}^2(\Omega_2)^2} \geqslant -\|\Psi_j\|_{\mathscr{L}^2(\Omega_2)^2}\left\|\left(\frac{\partial}{\partial x}\frac{\sqrt{\Delta}}{r^2+a^2}\right)\mathcal{A}_\kappa\Psi_j\right\|_{\mathscr{L}^2(\Omega_2)^2}$$
$$\geqslant -\frac{1}{2\varepsilon^2}\|\Psi_j\|^2_{\mathscr{L}^2(\Omega_2)^2} - \frac{\varepsilon^2}{2}\left\|\left(\frac{\partial}{\partial x}\frac{\sqrt{\Delta}}{r^2+a^2}\right)\mathcal{A}_\kappa\Psi_j\right\|^2_{\mathscr{L}^2(\Omega_2)^2}. \tag{22}$$

Hence, from (19), (21) and (22) we obtain

$$\|\mathcal{S}^{-1}H^{(\kappa)}\Psi\|^2 \geqslant \frac{1}{4}\sum_{j=1}^{2}\int_{\mathbb{R}}\int_0^\pi \left[\left\|\frac{\partial}{\partial x}\Psi_j(x,\theta)\right\|^2_{\mathbb{C}^2} + \frac{(1-2C^2\varepsilon^2)\Delta}{(r^2+a^2)^2}\|\mathcal{A}_\kappa\Psi_j(x,\theta)\|^2_{\mathbb{C}^2}\right]\mathrm{d}\theta\,\mathrm{d}x$$
$$- \frac{1}{2\varepsilon^2}\|\Psi\|^2_{\mathscr{L}^2(\Omega_2)^4} - \left\|\mathcal{S}^{-1}H_2^{(\kappa)}\right\|^2\|\Psi\|^2_{\mathcal{S}}.$$

If we choose ε small enough such that $2C^2\varepsilon^2<1$ and observe that $\|\Psi\|^2_{\mathscr{L}^2(\Omega_2)^4}\leqslant\|\mathcal{S}^{-1/2}\|^2\,\|\Psi\|^2_{\mathcal{S}}$, then we can choose $K>0$ large enough such that the estimate in the assertion holds.

For any $\Psi\in\mathcal{D}(H^{(\kappa)})$ there exists a sequence $(\Psi_n)_n$ such that

$$\Psi_n\to\Psi, \qquad H^{(\kappa)}\Psi_n\to H^{(\kappa)}\Psi$$

in $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)$. Note that for this sequence also the left-hand side of (17) converges by the dominated convergence theorem. Therefore, the assertion holds for all $\Psi\in\mathcal{D}(H^{(\kappa)})$. □

Note that the constant K does not depend on Ψ but only on the $\mathscr{L}_\infty$-bound C of $\sqrt{\Delta}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\sqrt{\Delta}}{r+a^2}$.

The next theorem shows that the partial wave operators $H^{(\kappa)}$ have the Rellich property.

Theorem 3.3. (Rellich property) *Let κ be any half integer, $R>0$ and set*

$$\Omega_{2,R} := (-R,R)\times(0,\pi). \tag{23}$$

Let $(\Psi_n)_n\subset\mathcal{D}(H^{(\kappa)})$ such that

$$\|\Psi_n\|_{\mathcal{S}} + \|H^{(\kappa)}\Psi_n\|_{\mathcal{S}} \leqslant K_0, \qquad n\in\mathbb{N},$$

for some constant $K_0>0$. Then there exists a subsequence $(\Psi_{n_\ell})_\ell$ and a $\Phi\in\mathscr{L}^2(\Omega_{2,R})^4$ such that

$$\Psi_{n_\ell}\to\Phi \qquad as \qquad \ell\to\infty$$

in $\mathscr{L}^2_{\mathcal{S}}(\Omega_{2,R})^4$.

Remark 3.4. As usual, we identify elements $\Psi\in\mathscr{L}^2(\Omega_2)^4$ with elements in $\mathscr{L}^2(\Omega_{2,R})^4$ by restriction. The assertion of theorem 3.3 implies that there exists a subsequence $(\Psi_{n_\ell})_\ell$ and a $\Phi\in\mathscr{L}^2(\Omega_2)^4$ such that $\chi_R\Psi_{n_\ell}\to\chi_R\Phi$ as $\ell\to\infty$ in $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)^4$ where χ_R is the characteristic function of $(-R,R)$.

Proof of theorem 3.3. Since the norms on $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)^4$ and $\mathscr{L}^2(\Omega_2)^4$ are equivalent, it suffices to consider convergence in the latter space. Let $\Psi_n =: \left(\psi_n^1, \psi_n^2\right)$ with $\psi_n^j \in \mathscr{L}^2(\Omega_2)^2$, $j = 1, 2$. Lemma 3.2 and the existence of a positive constant $\delta > 0$ such that

$$\frac{\Delta}{(r^2 + a^2)^2} \geqslant \delta, \qquad x \in (-R, R),$$

imply that there is a $K > 0$ such that

$$\sum_{j=1}^{2} \int_{-R}^{R} \int_0^{\pi} \left(\left\| \frac{\partial}{\partial x} \psi_n^j(x, \theta) \right\|^2_{\mathbb{C}^2} + \left\| \mathcal{A}_\kappa \psi_n^j(x, \theta) \right\|^2_{\mathbb{C}^2} \right) \mathrm{d}x\, \mathrm{d}\theta \leqslant K\left(\|\Psi_n\|^2_{\mathcal{S}} + \|H^{(\kappa)}\Psi_n\|^2_{\mathcal{S}}\right)$$
$$\leqslant K K_0^2, \tag{24}$$

for $n \in \mathbb{N}$. By lemma 2.3 the space $\mathscr{L}^2((0, \pi), \mathrm{d}\theta)^2$ has a basis of orthonormal eigenfunctions $\left(g_m^\kappa\right)_m$ of $\mathcal{A}_\kappa$ with corresponding eigenvalues λ_m^κ, $m \in \mathbb{Z} \setminus \{0\}$. Hence ψ_n^1 can be expanded in the double Fourier series

$$u_n = \sum_{\nu, m} \alpha_{\nu,m,n}\, \mathrm{e}^{-i\nu\pi x/R} g_m^\kappa(\theta),$$

$$\alpha_{\nu,m,n} = \frac{1}{2R} \int_{-R}^{R} \int_0^{\pi} \left\langle \psi_n^1(x, \theta), \mathrm{e}^{-\mathrm{i}\nu\pi x/R} g_m^\kappa(\theta) \right\rangle_{\mathbb{C}^2} \mathrm{d}x\, \mathrm{d}\theta$$

$$\sum_{\nu,m} |\alpha_{\nu,m,n}|^2 = \frac{1}{2R} \int_{-R}^{R} \int_0^{\pi} \left|\psi_n^1\right|^2 \mathrm{d}x\, \mathrm{d}\theta \leqslant \frac{1}{2R} \|\Psi_0\|^2_{\mathcal{S}}.$$

Moreover, inequality (24) yields

$$\sum_{\nu,m} \left[(\nu\pi/R)^2 + \left(\lambda_m^\kappa\right)^2\right] |\alpha_{\nu,m,n}|^2 \leqslant K K_0.$$

For fixed (ν, m) the sequence $(\alpha_{\nu,m,n})_n$ is a bounded sequence, hence it contains a subsequence such that $(\alpha_{\nu,m,n_\ell})_{n_\ell}$ is a Cauchy sequence for any (ν, m) by a diagonal series argument. Thus for arbitrary $L > 0$ it follows that

$$\begin{aligned}
\frac{1}{2R} \int_{-R}^{R} \int_0^{\pi} & \left|\psi_{n_\ell}^1 - \psi_{n_j}^1\right|^2 \mathrm{d}x\, \mathrm{d}\theta \\
&= \frac{1}{2R} \int_{-R}^{R} \int_0^{\pi} \left|\psi_{n_\ell}^1 - \psi_{n_j}^1\right|^2 \mathrm{d}x\, \mathrm{d}\theta = \sum_{\nu,m} |\alpha_{\nu,m,n_\ell} - \alpha_{\nu,m,n_j}|^2 \\
&= \sum_{(\nu\pi/R)^2 + (\lambda_m^\kappa)^2 \leqslant L} |\alpha_{\nu,m,n_\ell} - \alpha_{\nu,m,n_j}|^2 \\
&\quad + \sum_{(\nu\pi/R)^2 + (\lambda_m^\kappa)^2 \geqslant L} \frac{\left[(\nu\pi/R)^2 + \left(\lambda_m^\kappa\right)^2\right] |\alpha_{\nu,m,n_\ell} - \alpha_{\nu,m,n_j}|^2}{(\nu\pi/R)^2 + \left(\lambda_m^\kappa\right)^2} \\
&\leqslant \sum_{(\nu\pi/R)^2 + (\lambda_m^\kappa)^2 \leqslant L} |\alpha_{\nu,m,n_\ell} - \alpha_{\nu,m,n_j}|^2 + \frac{2KK_0}{L},
\end{aligned}$$

which shows that $(\psi_{n_\ell})_\ell$ is a Cauchy sequence in $\mathscr{L}^2(\Omega_2)^2$.

With the same argument we find that also the sequence $\left(\psi_{n_\ell}^2\right)_\ell$ contains a convergent subsequence in $\mathscr{L}^2(\Omega_2)^2$. □

4. Weak local energy decay

Since $H^{(\kappa)}$, $\kappa \in \mathbb{Z} + \frac{1}{2}$, is selfadjoint in $\mathscr{L}^2_{\mathcal{S}}(\Omega_2)^4$, it is the generator of a unitary group

$$U^{(\kappa)}(t) := \exp(-\mathrm{i}t\mathcal{S}^{-1}H^{(\kappa)}). \tag{25}$$

Theorem 4.1. *Let $\kappa \in \mathbb{Z} + \frac{1}{2}$, $R > 0$ and $\Psi_0 \in \mathcal{D}(H^{(\kappa)})$. For any sequence $(t_n)_n \subset \mathbb{R}$ there exist a subsequence $(t_{n_\ell})_\ell$ and $\Phi \in \mathscr{L}^2(\Omega_2)^4$ such that*

$$U^{(\kappa)}(t_{n_\ell})\Psi_0 \to \Phi \qquad \text{as} \qquad \ell \to \infty$$

in $\mathscr{L}^2(\Omega_2)^4$.

Proof. Let $\Psi_n := U^{(\kappa)}(t_n)\Psi_0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\Psi_n\|_{\mathcal{S}} + \|\mathcal{S}^{-1}H^{(\kappa)}\Psi_n\|_{\mathcal{S}} &= \|U^{(\kappa)}(t_n)\Psi_0\|_{\mathcal{S}} + \|\mathcal{S}^{-1}H^{(\kappa)}U^{(\kappa)}(t_n)\Psi_0\|_{\mathcal{S}} \\ &= \|U^{(\kappa)}(t_n)\Psi_0\|_{\mathcal{S}} + \|U^{(\kappa)}(t_n)\mathcal{S}^{-1}H^{(\kappa)}\Psi_0\|_{\mathcal{S}} \\ &= \|\Psi_0\|_{\mathcal{S}} + \|\mathcal{S}^{-1}H^{(\kappa)}\Psi_0\|_{\mathcal{S}}.\end{aligned}$$

Thus the assertion follows by theorem 3.3 with $K_0 = \|\Psi_0\| + \|H^{(\kappa)}\Psi_0\|$. □

Theorem 4.2. *For every $R > 0$ and $\Phi \in \mathscr{L}^2_{\mathcal{S}}(\Omega_2)$ the time mean of the localization of Φ in $(-R, R) \times (0, \pi)$ tends to zero:*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\int_{-R}^{R} \int_0^{\pi} \|U^{(\kappa)}(t)\Phi(x,\theta)\|^2_{\mathbb{C}^4} \mathrm{d}x\, \mathrm{d}\theta \right] \mathrm{d}t = 0.$$

Proof. It is well known that in the non-extreme Kerr–Newman case the operator $H^{(\kappa)}$ has no eigenvalues (Schmid 2004, Winklmeier and Yamada 2006, see also proposition 2.5). Let χ_R be the characteristic function of $\Omega_{2,R}$. Then the Rellich property (theorem 3.3) implies that $\chi_R(H^{(\kappa)} + \mathrm{i})^{-1}$ is a compact operator. Therefore, the assertion follows from the RAGE theorem, see Reed and Simon (1979). □

The decay of the partial waves implies the decay of the solutions of the original Cauchy problem (8):

Theorem 4.3. *Let $\Psi \in \mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$. Its Fourier expansion*

$$\Psi(x,\theta,\varphi) = \sum_{\kappa\in\mathbb{Z}+(1/2)} \mathrm{e}^{-\mathrm{i}\kappa\varphi}\Psi_\kappa(x,\theta)$$

converges strongly in $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$ and

$$\sum_{\kappa\in\mathbb{Z}+(1/2)} \|\Psi_\kappa(x,\theta)\|^2_{\mathcal{S}} < \infty.$$

Then

$$\widetilde{\Psi}(x,\theta,\varphi,t) = \sum_{\kappa\in\mathbb{Z}+(1/2)} \mathrm{e}^{-\mathrm{i}\kappa\varphi}U^{(\kappa)}(t)\Psi_\kappa(x,\theta)$$

is the unique weak solution of

$$\mathrm{i}\frac{\partial}{\partial t}\widetilde{\Psi} = \mathcal{S}^{-1}H\widetilde{\Psi}, \qquad \widetilde{\Psi}(0) = \Psi$$

(cf (8)) in $\mathscr{L}^2_{\mathcal{S}}(\Omega_3)^4$ and satisfies

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-R}^{R}\int_{0}^{\pi}\int_{0}^{2\pi}\|\mathcal{S}^{-1/2}\widetilde{\Psi}(x,\theta,\varphi,t)\|_{\mathbb{C}^4}^2\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}\varphi\right]\mathrm{d}t=0$$

for any $R > 0$.

Remark 4.4. The above results remain still valid if the electric potential eQr in (3) is substituted by any real-valued C^1-function $q(r)$ defined on $[r_+,\infty)$ such that $\lim_{r\to\infty}\frac{q(r)}{r}$ exists and $q'(r) = O(1)$ as $r\to\infty$.

Remark 4.5. The strong local energy decay follows from the absolute continuity of the spectrum of $\mathcal{S}^{-1}H_0$. This question was addressed by (Batic and Schmid 2006, lemma 5.3). However their proof is incomplete. We are currently trying to prove the absence of the singular continuous spectrum of $\mathcal{S}^{-1}H_0$.

5. Discussion

We have written the Dirac equation in the Kerr–Newman metric as a Cauchy problem and showed that the associated Dirac operator is selfadjoint in a certain Hilbert space (theorem 2.1). Therefore, the usual interpretation of the spectrum of the Dirac operator as in quantum mechanics is available. For example, eigenvalues would give rise to bound states. In the case of the non-extreme Kerr–Newman metric it is known that there are no eigenvalues in significant contrast to the extreme case where it is known that eigenvalues can exist. In general, however, it is not true that every bound state of a selfadjoint operator corresponds to an eigenvalue of the operator (see (Weidmann 2003)). However, since we could show that the partial wave operators of the Dirac operator have the Rellich property, the RAGE theorem yields that in the time average every Dirac particle must leave any given compact set outside the black hole. This implies that there are no time periodic solutions of the Dirac equation that could be interpreted as Dirac particles.

To show the decay of Dirac particles without averaging over time would require a proof of the absence of singular continuous spectrum. Until now, however, this problem seems to be unsolved. We conjecture that the absence of singular continuous spectrum can be proved by the so-called Mourre estimate which requires relatively hard analysis.

Acknowledgments

MW is grateful for the hospitality at Ritsumeikan University, Kusatsu. The research work on this paper started during her visit at Ritsumeikan University, Kusatsu, supported by Open Research Center Project for Private Universities: matching fund subsidy from MEXT, 2004–2008.

Appendix A. Equivalent representations of the time-independent Dirac operator

Remark A.1. If we do not apply the transformations (7) and (9) to the wavefunction $\widehat{\Psi}$, the time-independent Dirac equation takes the form

$$\mathrm{i}\frac{\partial}{\partial t}\widehat{\Psi} = \mathcal{S}^{-1}\widehat{\mathfrak{H}}\widehat{\Psi} \tag{A.1}$$

with $\widehat{\mathfrak{H}} = \widehat{\mathfrak{H}}_1 + \mathfrak{H}_2$ with $\mathfrak{H}_2$ as in theorem 2.1 and

$$\widehat{\mathfrak{H}}_1 = \begin{pmatrix} \widehat{\mathfrak{h}}_1 & \\ & -\widehat{\mathfrak{h}}_1 \end{pmatrix},$$

$$\widehat{\mathfrak{h}}_1 = -\sigma_3 \frac{\Delta}{r^2+a^2}\,\mathrm{i}\frac{\partial}{\partial r} - \frac{\sqrt{\Delta}}{r^2+a^2}\left(\sigma_1 \mathrm{i}\left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) + \sigma_2 \frac{1}{\sin\theta}\,\mathrm{i}\frac{\partial}{\partial\varphi}\right).$$

$\widehat{\mathfrak{H}}$ is formally symmetric in the Hilbert space

$$\widehat{\mathscr{L}}^2_{\mathcal{S}} := \mathscr{L}^2((r_+,\infty)\times(0,\pi)\times[0,2\pi))$$

with the scalar product

$$(\widehat{\Psi},\widehat{\Phi})^{\wedge}_{\mathcal{S}} = \int_{r_+}^{\infty}\int_0^{\pi}\int_0^{2\pi} \langle \widehat{\Psi}(r,\theta,\varphi), \mathcal{S}(r,\theta,\varphi)\widehat{\Phi}(r,\theta,\varphi)\rangle_{\mathbb{C}^4} \frac{\Delta}{r^2+a^2}\,\mathrm{d}r \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Remark A.2. In theorem 2.1 we have equipped the Hilbert space $\mathscr{L}^2((-\infty,\infty)\times(0,\pi)\times(0,2\pi))$ with a scalar product $(\cdot,\cdot)_{\mathcal{S}}$ such that $\mathfrak{H}$ is formally symmetric. Alternatively, we can work with the usual scalar product, but then we have to define the Dirac operator in the following way:

$$\widetilde{H}_0\Psi = \mathcal{S}^{-1/2}\mathfrak{H}\mathcal{S}^{-1/2}\Psi, \qquad \mathcal{D}(\widetilde{H}_0) = \mathscr{C}_0^{\infty}((-\infty,\infty)\times(0,\pi)\times[0,2\pi))^4$$

on $\mathscr{L}^2((-\infty,\infty)\times(0,\pi)\times[0,\pi))^4$ equipped with the usual scalar product

$$(\Psi,\Phi)^{\sim} = \int_{\mathbb{R}}\int_0^{\pi}\int_0^{2\pi} \langle\Psi(x,\theta,\varphi),\Phi(x,\theta,\varphi)\rangle_{\mathbb{C}^4}\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Note that $\mathcal{S}$ leaves the space $\mathscr{C}_0^{\infty}((-\infty,\infty)\times(0,\pi)\times[0,2\pi))^4$ invariant.

Appendix B. Rellich's theorem

Theorem B.1 (Rellich's theorem). *Let $d \in \mathbb{N}$, $K \subseteq \mathbb{R}^d$ compact and Ω a bounded open neighborhood of K. Further let $s > t \in \mathbb{N}$. Then for each $c > 0$ the set*

$$F := \{u \in H^s(\mathbb{R}^d) : \|u\|_{H^s(\mathbb{R}^d)} \leqslant c, \operatorname{supp}(u) \subseteq K\}$$

is precompact in $H^t(\Omega)$, i.e., for each sequence $(f_n)_n \subset F$ there exists an $f_0 \in H^t(\mathbb{R}^d)$ and a subsequence $(f_{n_k})_k$ such that $f_{n_k} \to f_0$ in $H^t(\Omega)$.

Proof. See, e.g., Mizohata (1973). □

References

Batic D and Schmid H 2006 The Dirac propagator in the Kerr–Newman metric *Prog. Theor. Phys.* **116** 517–44

Belgiorno F and Cacciatori S 2008 The Dirac equation in Kerr–Newman–AdS black hole background arXiv:0803.2496v3

Belgiorno F and Martellini M 1999 Quantum properties of the electron field in Kerr–Newman black hole manifolds *Phys. Lett.* B **453** 17–22

Chandrasekhar S 1976 The solution of Dirac's equation in Kerr geometry *Proc. R. Soc. Lond.* A **349** 571–5

Chandrasekhar S 1998 *The Mathematical Theory of Black Holes (Oxford Classic Texts in the Physical Sciences)* (New York: Oxford University Press) (reprint of 1992 edition)

Finster F, Kamran N, Smoller J and Yau S T 2000 Nonexistence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry *Commun. Pure Appl. Math.* **53** 902–29

Finster F, Kamran N, Smoller J and Yau S T 2003 The long-time dynamics of Dirac particles in the Kerr–Newman black hole geometry *Adv. Theor. Math. Phys.* **7** 25–52

Mizohata S 1973 *The Theory of Partial Differential Equations* (New York: Cambridge University Press) (translated from Japanese by Katsumi Miyahara)

Page D N 1976 Dirac equation around a charged, rotating black hole *Phys. Rev.* D **14** 1509–10

Reed M and Simon B 1979 *Methods of Modern Mathematical Physics: III. Scattering theory* (New York: Academic Press)

Schmid H 2004 Bound state solutions of the Dirac equation in the extreme Kerr geometry *Math. Nachr.* **274/275** 117–29

Weidmann J 1987 *Spectral Theory of Ordinary Differential Operators (Lecture Notes in Mathematics* vol 1258*)* (Berlin: Springer)

Weidmann J 2003 *Lineare Operatoren in Hilbertraumen, Teil II* (Stuttgart: Teubner Verlag)

Winklmeier M 2006 The angular part of the Dirac equation in the Kerr–Newman metric: estimates for the eigenvalues *PhD Thesis* Universität Bremen

Winklmeier M and Yamada O 2006 Spectral analysis of radial Dirac operators in the Kerr–Newman metric and its applications to time-periodic solutions *J. Math. Phys.* **47** 102503